Math 275D Lecture 22 Notes

Daniel Raban

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1 Local Martingales for Stochastic Integrals

1.1 Stochastic integrals as local martingales

Example 1.1. Let $X_t = B_{t \wedge \tau}$, where $\tau = \inf\{t : B_t \leq -1\}$. Then define

$$Y_t = \begin{cases} X_{\frac{t}{1-t}} & t \leq 1 \\ -1 & t \geq 1. \end{cases}$$

Then Y_t is a local martingale with

$$\tau_n = \begin{cases} \inf\{t : Y_t \geq n\} & \text{if } \inf < \infty \\ n & \text{otherwise.} \end{cases}$$

We need to check that $Y_t^{(n)} := Y_{t \wedge \tau_n}$ is a martingale. First, $Y_t^{(n)} \in \mathbb{R}$, for each t, so this is well-defined. Then $Y_t^{(n)} = B_{\widetilde{t} \wedge \widetilde{\tau}_n} = B_{\frac{t}{1-t} \wedge \widetilde{\tau}_n}$, where $\widetilde{\tau}_n = \frac{\tau_n}{1-\tau_n}$ and $\widetilde{\tau}_n = \inf\{\widetilde{t} : B_{\widetilde{t}} = n \text{ or } -1\}$.

If $f \in L^2_{\text{loc}}([0,T])$, why is $\int_0^t f \, dB_s$ a local martingale? We know that $\int_0^T f^2 \, dt < \infty$ a.s., and $F_t = F_T$ if $t > T$. Let

$$\tau_n = \begin{cases} \inf\{t : \int_0^t f^2 \geq n\} & \text{if } \inf < \infty \\ nT & \text{otherwise.} \end{cases}$$

1.2 Recovering martingales from local martingales

What properties do local martingales have?

Example 1.2. Let X_t be a local martingale with $X_0 = 0$, and let $\tau = \inf\{t : X_t = -a \text{ or } b\}$. We want to calculate $\mathbb{P}(X_\tau = a)$. When this is a martingale, we use the fact that $\mathbb{E}[X_\tau] = 0$.

Define $X_t^{(n)} := X_{t \wedge \tau_n}$, which is a martingale. Then $X_{t \wedge \tau}^{(n)}$ is a (bounded) martingale. Then $\mathbb{E}[X_\tau^{(n)}] = 0$. This gives us $\mathbb{E}[X_{\tau \wedge \tau_n}] = 0$. But $X_\tau = \lim_{n \to \infty} X_{\tau \wedge \tau_n}$ as $n \to \infty$. Since this is pointwise bounded convergence, we get L^1 convergence: $\mathbb{E}[X_{\tau \wedge \tau_n}] = \mathbb{E}[X_t]$.

Theorem 1.1. *Let X_t be a local martingale with a sequence of stopping times τ_n, and let τ be a stopping time. Then $Y_t := X_{t \wedge \tau}$ is a local martingale. Furthermore, if X_t is bounded ($\sup_{\omega,t} |X_t| \leq M$), then X_t is a martingale.*

Proof. The martingale property is $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$. For local martingales, we have $\mathbb{E}[X_{t \wedge \tau_n} \mid \mathcal{F}_s] = X_{s \wedge \tau_n}$. Letting $n \to \infty$ so $\tau_n \to \infty$, the right hand side becomes $X_{s \wedge \tau_n}$. However, this does not necessarily mean that the left hand side goes to $\mathbb{E}[X_t \mid \mathcal{F}_s]$; we can have $f_n \to f$ but $\int f_n \not\to \int f$. But Fatou's lemma tells us that if $X_t \geq 0$,

$$\mathbb{E}[X_t \mid \mathcal{F}_s] \leq \liminf_n \mathbb{E}[X_{t \wedge \tau} \mid \mathcal{F}_s] \leq \lim_n X_{s \wedge \tau_n} \leq X_s.$$

So if $X_t \geq 0$, then X_t is a supermartingale. So if $|X_t| \leq M$, then X_t is a martingale. $\square$

1.3 Local martingale form of Itô's formula

We have learned one form of Itô's formula:

$$f(B_t) - f(0) = \int_0^t f'(B_s)\, dB_s + \frac{1}{2} \int_0^t f''(B_s)\, ds$$

for $f \in C^2$.

People like to use the formula in another way:

$$\int_0^t f(B_s)\, dB_s = \int_0^{B_t} f(s)\, ds - \frac{1}{2} \int_0^t f'(B_s)\, ds$$

for $f \in C^1$. Here, the right hand side depends on ω, so the left hand side should, as well. So the left hand side is determined "ω by ω" or "pathwise."

Let's extend this notion to $f(t, B_t)$ for $f \in C^1 \times C^2$. We have

$$f(t, B_t) = f(0, B_0) + \int_0^t f^{(1,0)}(s, B_s)\, ds + \int_0^t f^{(0,1)}(s, B_s)\, dB_s + \frac{1}{2} \int_0^t f^{(0,2)}(s, B_s)\, dB_s,$$

where the superscripts denote partial derivatives. Here, if $f^{(1,0)} = f^{(0,2)}$, then $f(t, B_t)$ equals a constant plus the 3rd term, which is a local martingale. This will give us that f is a local martingale.